Arthur Cayley (August 16, 1821 - January 26, 1895) was a British Mathematician and Founder of the Modern British School of Pure Mathematics. As a child, Cayley enjoyed solving complex math problems for amusement. At eighteen, he entered Trinity College, Cambridge, where he excelled in Greek, French, German, and Italian, as well as in Mathematics. He worked as a lawyer for 14 years. He was consequently able to prove the Cayley-Hamilton Theorem that every square matrix is a root of its own characteristic polynomial. He was the first to define the concept of a group in the modern way as a ‘set with a binary operation satisfying certain laws’.

DETERMINANTS

In algebra, a determinant is a function depending on \( n \) that associates a scalar, \( \det(A) \), to every \( n \times n \) square matrix \( A \). The fundamental geometric meaning of a determinant acts as the scale factor for volume when \( A \) is regarded as a linear transformation. Determinants are important both in Calculus, where they enter the substitution rule for several variables, and in Multilinear Algebra.

For a fixed positive integer \( n \), there is a unique determinant function for the \( n \times n \) matrices over any commutative ring \( R \). In particular, this function exists when \( R \) is the field of real or complex numbers.

The determinant of a square matrix \( A \) is also sometimes denoted by \( |A| \). This notation can be ambiguous since it is also used for certain matrix norms and for the absolute value. However, often the matrix norm will be denoted with double vertical bars e.g., \( \| A \| \) and may carry a subscript as well. Thus, the vertical bar notation for determinant is frequently used (e.g., Cramer’s rule and minors).

\[
A = \begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i \\
\end{bmatrix}
\]

For example, for matrix

\[
\begin{vmatrix}
a & b & c \\
d & e & f \\
g & h & i \\
\end{vmatrix}
\]

the determinant \( \det(A) \) might be indicated by \( |A| \) or more explicitly as
MATRICES

“Matrix is a rectangular array of elements in rows and columns put in a large braces ” – defines the lexicon. But there is more than that meets the eye. The term “matrix” was coined in 1848 by J.J. Sylvester. Arthur Cayley, William Rowan Hamilton, Grassmann, Frobenius and von Neumann are among the famous mathematicians who have worked on matrix theory. Though it has no numerical value as a whole, it is put to use in myriad fields. Matrix represents transformations of coordinate spaces. It is a mathematical shorthand to help study problems of entries. It provides convenient and compact notation for representation of data. Out of the inexhaustive uses of matrices the following may be called as the predominant:

1. Finding sets of solutions of a system of linear equations.
2. To study the relation on sets, directed routes and cryptography i.e. coding and decoding secret messages.
3. Used in input output analysis of industries to test the viability of the economic systems of industries.
4. Used in finite element methods (Civil and Structural engineering) and in network analysis (Electrical and Electronic engineering).
5. For handling large amount of data in computers, which is widely useful at present.
6. It is widely used in other fields like Statistics, Psychology, Operation Research, Differential Equations, Mechanics, Electrical Circuits, Nuclear Physics, Aerodynamics, Astronomy, Quantum Mechanics etc.,

The students are already acquainted with the basic operations of Matrices such as matrix additions, matrix multiplication, etc. However, some of these important properties are now recalled to participate further learning.

1. Matrix addition is commutative and associative.
   ie. i) A + B = B + A     ii) A + (B + C) = (A + B) + C

2. Matrix multiplication is not commutative. ie. AB ≠ BA

3. But Matrix multiplication is associative. ie. A(BC) = (AB)C

4. Distributive Property
   i) A(B + C) = AB + AC    (Left Distributive Property of Matrix multiplication over addition)
   ii) (A + B)C = AC + BC   (Right Distributive Property of Matrix multiplication over addition)

Note : i) \((A+B)^2 \neq A^2 + 2AB + B^2\)  In fact \((A+B)^2 = A^2 + AB + BA + B^2\)
Important Points in Matrices & Determinants

Properties of the Determinants (Without Proof)

1) The value of the Determinant is not altered by interchanging the rows and columns (It is symbolically denoted as $R \leftrightarrow C$).

\[
\begin{vmatrix}
    a_1 & a_2 & a_3 \\
    b_1 & b_2 & b_3 \\
    c_1 & c_2 & c_3 \\
\end{vmatrix}
\quad \text{&} \quad
\begin{vmatrix}
    a_1 & a_2 & a_3 \\
    b_3 & b_2 & b_1 \\
    c_1 & c_2 & c_3 \\
\end{vmatrix}
\Rightarrow \Delta = \Delta_1
\]

(i.e)

2) If any two rows or any two columns of a determinant are interchanged then the value of the determinant changes in sign, but its numerical value is unaltered

Example

\[
\begin{vmatrix}
    a_1 & b_1 & c_1 \\
    a_2 & b_2 & c_2 \\
    a_3 & b_3 & c_3 \\
\end{vmatrix}
\quad \Delta_1 = \begin{vmatrix}
    b_1 & a_1 & c_1 \\
    b_2 & a_2 & c_2 \\
    b_3 & a_3 & c_3 \\
\end{vmatrix}
\Delta_1 = -\Delta
\]

(Here $C_1$ & $C_2$ are interchanged represented, as $C_1 \leftrightarrow C_2$)

Note: If a row (or a column) is made to pass through two parallel rows (or column) then the value of the determinant remains unchanged ie.

\[
\Delta_1 = (-1)^2 \Delta \quad \Rightarrow \Delta_1 = \Delta.
\]

In general if a row (or a column) is made to pass through ‘$n$’ such parallel rows

(or columns) then we have the new determinant $\Delta_1 = (-1)^n \Delta$.

3) If any two rows (or two columns) are identical then the value of the determinant is zero

Example:

\[
\begin{vmatrix}
    a_1 & b_1 & c_1 \\
    a_2 & b_2 & c_1 \\
    a_3 & b_3 & c_3 \\
\end{vmatrix}
\Delta = 0 \quad (\text{Since } R_1 \text{ & } R_2 \text{ are identical})
\]

4) If every element in a row or a column of a determinant is multiplied by a non-zero constant ‘$k$’ (i.e $k \neq 0$), then the value of the determinant is multiplied by $k$.

Example :

\[
\begin{vmatrix}
    a_1 & b_1 & c_1 \\
    a_2 & b_2 & c_1 \\
    a_3 & b_3 & c_3 \\
\end{vmatrix}
\Delta = \begin{vmatrix}
    ka_1 & kb_1 & kc_1 \\
    a_2 & b_2 & c_2 \\
    a_3 & b_3 & c_3 \\
\end{vmatrix}
\Delta = k\Delta
\]

(i.e) $\Delta_1 = k\Delta$ (Every element of $R_1$ is multiplied by $k$)

Note: If all the three rows (or all the three columns) of a 3$^{rd}$ order determinant are multiplied by $k$, then the value of the determinant is multiplied by

$k^3 (k \times k \times k = k^3)$ ;
(ie. $\Delta_1 = k^3\Delta$)

5) If every entry in a row (or column) can be expressed as sum of two quantities then the given determinant can be expressed as sum of two determinants of the same order

$$\Delta = \begin{vmatrix} a_1 + a_2 & b_1 + b_2 & c_1 + c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

i.e.

**Note:** Two determinants of the same order can be added, by adding the corresponding entries of a particular row (or a column), provided the other two rows (or the columns) in both the determinants are the same.

**Factor Theorem**

1) If two rows (or columns) of a determinant $\Delta$ are identical on putting $x = a$, then $(x - a)$ is a factor of $\Delta$

2) If all the three rows (or columns) of a determinant $\Delta$ are identical on putting $x = a$, then $(x - a)^2$ is a factor of $\Delta$

**Theorem**

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Let $A_1, B_1, C_1$ be the cofactors of $a_1, b_1, c_1$ in $\Delta$. Then **cofactor determinant**.

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^2$$

**Note:**

1) $a_1A_1 + b_1B_1 + c_1C_1 = a_2A_2 + b_2B_2 + c_2C_2 = a_3A_3 + b_3B_3 + c_3C_3 = \Delta$

2) $a_1A_1 + b_1B_2 + c_1C_2 = 0$ ; $a_2A_2 + b_2B_2 + c_2C_2 = 0$

6) A determinant is unaltered when to each element of any row (or column) the equimultiples of any corresponding row (or column) are added

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + mb_1 + nc_1 & b_1 & c_1 \\ a_2 + mb_2 + nc_2 & b_2 & c_2 \\ a_3 + mb_3 + nc_3 & b_3 & c_3 \end{vmatrix}$$

i.e. we have $\Delta = \Delta_1$

**Note:** Usual operations (Elementary transformations) that are carried out so that the value of the determinant is unaltered are

1) $C_1 \leftrightarrow C_1 - C_2$; $C_2 \leftrightarrow C_2 - C_3$  [This is done if every entry in a row is the same]
2) \[ R_1 \leftrightarrow R_1 - R_2, \quad R_2 \leftrightarrow R_2 - R_3 \] [This is done if every entry in a column is the same]

3) \[ C_1 \leftrightarrow C_1 + C_2 + C_3 \] [This is done if every entry after addition in first column becomes the same (or has a common factor)]

4) \[ R_1 \leftrightarrow R_1 + R_2 + R_3 \] [This is done if every entry after addition in first row becomes the same (or has a common factor)]

Matrix and its Applications

Matrices are rectangular arrangements of numbers in rows and columns put within a large paranthesis. Matrices are denoted by capital letters like \( A, B, C \) and so on.

The order of a Matrix is the number of rows and the number of columns that are present in a Matrix. Suppose a Matrix \( A \) has \( m \) rows and \( n \) columns, the order of Matrix \( A \) is denoted by \( m \times n \) read as \( m \) by \( n \).

The order of the above said Matrix \( A \) is \( 3 \times 3 \). The order of the above said Matrix \( B \) is \( 2 \times 3 \).

Difference between Matrix and a Determinant

1. Matrices do not have definite value, but determinants have definite value.

2. In a Matrix the number of rows and columns may be unequal, but in a Determinant the number of rows and columns must be equal.

3. The entries of a Matrix are listed within a large paranthesis (large braces), but in a determinant the entries are listed between two strips (i.e. between two vertical lines).

4. Let \( A \) be a Matrix. Matrix \( kA \) is obtained by multiplying all the entries of the Matrix by \( k \).

Let \( \begin{vmatrix} A \end{vmatrix} \) be any Determinant. \( k \begin{vmatrix} A \end{vmatrix} \) is obtained by multiplying ‘every entry of a row’ or ‘every entry of a column’ by \( k \) (i.e. in order to multiply a Determinant by a number \( k \)
every entry of a row or every entry of a column is multiplied by \( k \).

**Caution :** Do not multiply all the entries of the Determinant by \( k \) in order to multiply the
Determinant by \( k \).

**Note :** If \( A \) is a 3\(^{rd} \) order square matrix\( |kA| = k^3 |A| \)

In general if \( A \) is an \( n \)\(^{th} \) order square matrix\( |kA| = k^n |A| \)

1. **Adjoint of a Matrix :** Let \( A = [a_{ij}] \) be a square matrix of order \( n \). Let \( A_{ij} \) be the
cofactor entry of each \( a_{ij} \) of the Matrix \( A \). Then \( [A_{ij}]^T \) is the adjoint of the Matrix \( A \).

i.e \( \text{Adjoint of matrix } 'A' \text{ is the transpose of co-factor matrix } A \)

\[ \text{Adj } A = [A_{ij}]^T \; \text{ where } A_{ij} \text{ is the co-factor entry of matrix } A \]

**Note :** Adjoint of a matrix can be found for square matrices only and we have
\[ A(\text{Adj } A) = (\text{Adj } A)A = |A|I \]

**Inverse :** Let \( A \) be a square matrix of order \( n \). Then a matrix \( B \), is called the inverse of
matrix \( A \) if, \( AB = BA = I_n \) (unit matrix of order \( n \)). The inverse of \( A \) is denoted by \( A^{-1} \)

The formula for finding the inverse of matrix \( A \) is denoted by

\[ A^{-1} = \frac{\text{adj}(A)}{|A|} \]

**Caution :** If \(|A| = 0 \) \( \Rightarrow \) the inverse of matrix \( A \) does not exist.
But the adjoint of matrix \( A \) exists.

**Singular Matrix :** A square matrix ‘\( A \)’ of order ‘\( n \)’ is a singular matrix if its deter-
minant value is zero. i.e. \(|A| = 0 \) \( \Rightarrow \) matrix \( A \) is a singular matrix. Inverse does not
exist for a singular matrix

**Non-Singular Matrix :** A square matrix ‘\( A \)’ of order \( n \) is a non-singular matrix if
its determinant value is not equal to zero. i.e. \(|A| \neq 0 \) \( \Rightarrow \) matrix \( A \) is a non-singular
matrix. Inverses do exist for non-singular matrices.

i.e. If \( A \) is a non-singular square matrix then \( B \) is called the inverse of \( A \), if
Note: Inverse for a matrix exists only for a square matrix, provided its determinant value is not equal to zero.

Note: 1. \((AB)^{-1} = B^{-1}A^{-1}\)  This is known as inverse reversal law
2. \((AB)^{r} = B^{r}A^{r}\)
3. \((A^{r})^{-1} = (A^{3})^{r}\)
4. If \(A\) is a square matrix of order ‘\(r\)’, then \(\text{adj}A - A^{n-1}\)

**Rank of a Matrix** : Matrix \(A\) is said to be of rank ‘\(r\)’, if
i) \(A\) has at least one minor of order ‘\(r\)’ which does not vanish.
ii) Every minor of \(A\) of order \((r + 1)\) and higher order vanishes.

In other words Rank of Matrix \(A\) is equal to the order of the highest non-vanishing minor of the matrix.

Note: Rank of a Matrix is less than or equal to the least of its row or its column.

\[\begin{align*}
\text{i.e.} & \quad 1. \text{If order of matrix } A \text{ is } 3 \times 3 \quad \rho(A) \leq 3 \\
& \quad 2. \text{If order of matrix } A \text{ is } 5 \times 4 \quad \rho(A) \leq 4 \\
& \quad 3. \text{If order of matrix } A \text{ is } 2 \times 3 \quad \rho(A) \leq 2
\end{align*}\]

**Echelon Form**

Finding the rank of a matrix involves more computation work. Reducing it into the Echelon form may be useful in finding rank.

**Echelon form of a matrix** :
A matrix \(A\) (of order \(m \times n\)) is said to be in echelon form (triangular form) if
i) Every row of \(A\) which has all its entries 0 occurs below every row which has a non-zero entry.
ii) The first non-zero entry in each non-zero row is 1.
iii) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

By elementary operations one can easily bring the given matrix to its echelon form.
Equivalent matrices:
Two matrices A and B of the same order are said to be equivalent if one can be obtained from the other by the applications of a finite number of sequences of elementary transformation. “Matrix A is equivalent to matrix B” is symbolically denoted by $A \sim B$.

Note: Equivalent matrices have the same rank.

Solution of Non homogeneous equations:

Let us define $\Delta$, $\Delta_x$, $\Delta_y$, and $\Delta_z$ as:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$\Delta$ is a determinant obtained by listing the co-efficients of $x,y,z$ of the above system of equations in the given order.

$$\Delta_x = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$\Delta_x$ is a determinant obtained by replacing the first column entries of $\Delta$ by the constant terms $b_1, b_2, b_3$ listed on the right hand side of the system of equations.

$$\Delta_y = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

$\Delta_y$ is a determinant obtained by replacing the second column entries of $\Delta$ by the constant terms $b_1, b_2, b_3$ listed on right hand side of the system of equations.

$$\Delta_z = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

$\Delta_z$ is a determinant obtained by replacing the third column entries of $\Delta$ by the constant terms $b_1, b_2, b_3$ listed on right hand side of the system of equations.

Consistent / Inconsistent

1. The given system of equations is said to be consistent if the system of linear equations possesses at least one solution.

The given system of equations is said to be inconsistent if the system of linear equations has no solution.

2. i) If $\Delta \neq 0$ the system is consistent with a unique solution and the solution may be
obtained using formula

\[ x = \frac{\Delta x}{\Delta}; \quad y = \frac{\Delta y}{\Delta}; \quad z = \frac{\Delta z}{\Delta} \]

ii) If \( \Delta = 0 \) the system may be consistent or inconsistent

The following table gives a clear idea of the system of linear equations being consistent or inconsistent if \( \Delta = 0 \)

<table>
<thead>
<tr>
<th>Consistent</th>
<th>Inconsistent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. also ( \Delta_x = \Delta_y = \Delta_z = 0 ) and atleast one of the</td>
<td>but it atleast one of ( \Delta_x, \Delta_y ) or ( \Delta_z ) is non-zero ⇒ the system has no solution</td>
</tr>
<tr>
<td>2 x 2 minors of ( \Delta ) is non-zero ⇒ the system has infinite solutions; put ( z = k ) and solve any two equations</td>
<td>⇒ the system has no solution</td>
</tr>
<tr>
<td>2. also ( \Delta_x = \Delta_y = \Delta_z = 0 ); all 2 x 2 minors of ( \Delta, \Delta_x, \Delta_y ) and ( \Delta_z ) are zero. If ( \Delta ) has atleast one non-zero element, the system has infinite solutions. Put ( y = s, \ z = t ) and solve any one equation.</td>
<td>also ( \Delta_x = \Delta_y = \Delta_z = 0 ); all 2 x 2 minors of ( \Delta ) are zero but atleast one of the 2 x 2 minors of ( \Delta_x, \Delta_y ) or ( \Delta_z ) is non-zero ⇒ the system has no solution</td>
</tr>
</tbody>
</table>

The description displayed in the above table is neatly represented in the following diagram of equations to be consistent or inconsistent.

\( \Delta \neq 0 \) ⇒ The system is consistent with a unique solution. In other words the three planes have only one common point. i.e. S is the only common point as shown in the figure.

\( \Delta = 0 \) and \( \Delta_x = \Delta_y = \Delta_z = 0 \) and if one of the 2 x 2 minors of \( \Delta \) is not equal to zero ⇒ The system is consistent with infinite no. of solutions. In this case the three planes intersect along a line as shown.
\[ \Delta = 0 \text{ and } \Delta_x = \Delta_y = \Delta_z = 0 \] and all the 2 x 2 minors of \( \Delta, \Delta_x, \Delta_y, \Delta_z \) are also zero \( \Rightarrow \) The system is consistent with infinite no. of solutions.

i.e. all the three equations represent one and the same plane. The three planes are coincident.

\[ \Delta = 0 \text{ and } \Delta_x = \Delta_y = \Delta_z = 0 \text{ and all } 2 \times 2 \text{ the minors of } \Delta \text{ are also zero and if one of the } 2 \times 2 \text{ minors of } \Delta_x, \Delta_y, \Delta_z \text{ is not equal to zero } \Rightarrow \text{ The system is inconsistent and has no solution. } \]

i.e. The three planes are parallel planes.

**Consistency or Inconsistency using rank**

Consider the system of linear equations

\[ a_{11}x + a_{12}y + a_{13}z = b_1 \\
\]

\[ a_{21}x + a_{22}y + a_{23}z = b_2 \\
\]

\[ a_{31}x + a_{32}y + a_{33}z = b_3 \\
\]

Take matrix \( A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \); matrix \( B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \) and matrix \( X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \)

The Augmented matrix \([A, B] = \)

\[ \begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{pmatrix} \]

i) Write down the given system of equations in the form of a matrix equation \( AX = B \)
ii) Find the augmented matrix \([A, B]\) of the system of equations.

iii) Find the rank of \(A\) and rank of \([A, B]\) by applying only elementary row operations.

Caution: Elementary Column operations should not be applied in augmented matrix

iv) a) If the rank of \(A\) \(\neq\) rank of \([A, B]\) then the system is inconsistent and has no solution.

b) If the rank of \(A\) = rank of \([A, B]\) = \(n\), where \(n\) is the number of unknowns in the system, then \(A\) is a non-singular matrix and the system is consistent and it has a unique solution.

c) If the rank of \(A\) = rank of \([A, B]\) < \(n\), then also the system is consistent, but has an infinite number of solutions.

Homogeneous Linear System

\[
\begin{align*}
    a_1x + b_1y + c_1z &= 0 \\
    a_2x + b_2y + c_2z &= 0 \\
    a_3x + b_3y + c_3z &= 0
\end{align*}
\]

is known as a system of Homogeneous Linear Equation in three unknowns.

This system has a trivial solution. i.e. \(x = 0\), \(y = 0\), \(z = 0\) is the trivial solution.

If this system has any other solutions in addition to the trivial solutions then they are known as non-trivial solutions.

In the case of Homogeneous Linear System of equations we have only two possibilities

i) The system has only a trivial solution

ii) The system has infinitely many solutions in addition to the trivial solution, known as non-trivial solutions.

Here Matrix \(A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\) and Matrix \(B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\)